estimates (8.11) – (8.15) are preserved. The theorem is completely proved.


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Translated by N.H.C.

UDC 531.31

# DYNAMIC SYSTEMS ARISING ON THE INVARIANT TORI OF THE KOWALEWSKA PROBLEM

PMM Vol. 39, № 1, 1975, pp. 24-29

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(Received April 1, 1974)

We determine the gyration numbers of the dynamic systems arising on the two-dimensional invariant tori in Kowalewska's problem. We have shown that they equal the ratio of the periods of a hyperelliptic integral containing the Kowalewska polynomial. Using the general theorem on the reduction of equations on an $n$-dimensional torus, proved in the paper, the differential equations on the two-dimensional invariant tori mentioned are reduced by an invertible change of variables to the form $\varphi_i = \omega_i$ where $\omega_i = \text{const}$, $i = 1, 2$. We prove also that in the case of rapid rotations of the body the combined levels of the four first integrals of the problem consist of two tori; the dynamic systems arising on these tori are isomorphic.

**1. Remarks on the topological properties of the combined levels of first integrals.** The Euler-Poisson equations of the problem of the motion of a heavy rigid body around a fixed point form an analytic system of differential equations defined in $R^6$ $\{x : pqr\gamma_1\gamma_2\gamma_3\}$. There is an integral invariant in this system, whose density $M(x) \equiv 1$ (i.e. the phase volume is invariant relative to a one-parameter group $g^t$ of shifts along the trajectories of the Euler-Poisson equations). These equations always have three algebraic first integrals: the energy integral $(H)$, the area integral $(L)$ and the geometric integral $(\Gamma)$. If the rigid body is a Kowalewska top, then there exists a fourth algebraic integral $K$.

By $E$ we denote the following set:

$$E = \{x : H = 6h,\ L = 2l,\ \Gamma = 1,\ K = k^2\}\ (E \subset R^6)$$

It is compact, since the set $\{H = 6h,\ \Gamma = 1\}$ is bounded in $R^6$ and $E$ is closed.

It is clear that $E$ is invariant relative to group $g^t$. It is obvious that those values of parameters $6h$, $2l$, $k^2$, for which the first integrals depend upon $E$, form a set of measure zero. Everywhere below we consider only such sets $E$ on which the first integrals are independent. In this case $E$ is a smooth two-dimensional manifold.

We denote the restriction of group $g^t$ onto $E$ by $g_E{}^t$. From Jacobi's theorem on the last factor (see [1]) follows the existence on $E$ of a Jordan measure $\nu(x)$ invariant relative to $g_E{}^t$. Consequently, the triplet $(E, g_E{}^t, \nu)$ is a classical dynamic system (see [2] for the definition). The task of the present paper is the study of such systems.

At first we investigate the topological properties of manifold $E$. There are no singular points of the system of Euler-Poisson differential equations on $E$. In fact, singular points correspond to steady-state rotations (or to relative equilibria) of the body. But, as proved in [3] the integrals of energy and moment in these solutions are related. We have stipulated that such cases are not to be examined here.

Manifold $E$ is orientable. Hence, each connected component of $E$ is a two-dimensional torus (as in every connected orientable compact two-dimensional manifold admitting of a tangent vector field without singular points; for example, see [2]). It is not difficult to prove that for small values of parameter $\mu$ – the product of the body's weight and of the distance from the center of gravity to the suspension point – the manifold $E$ consists of two connected components.

**2. Calculation of the gyration numbers.** Authors of papers devoted to Kowalewska's problem have used equations of motion in the Kowalewska variables $s_1$, $s_2$, wherein complex quantities enter explicitly [1]. This gives rise to specific inconveniences when investigating the real motions of he system. The imaginary quantities can be avoided by writing the equations of motion in the following form:

$$\frac{ds_1}{\sqrt{-\Phi(s_1)}}+\frac{ds_2}{\sqrt{-\Phi(s_2)}}=0,\qquad \frac{s_1ds_1}{\sqrt{-\Phi(s_1)}}+\frac{s_2ds_2}{\sqrt{-\Phi(s_2)}}=\frac{dt}{2} \tag{2.1}$$

$$\Phi(z)=\{z[(z-3h)^2+\mu^2-k^2]-2\mu^2l^2\}(z-3h-k)(z-3h+k)$$

Let us prove that in a real motion the variables $s_1$ and $s_2$ take real values. To do this we write out the formulas, due to Kowalewska, which express $s_1$ and $s_2$ in terms of the Euler-Poisson variables $(pqr\gamma_1\gamma_2\gamma_3)$ [1]

$$s_{1,2}=3h+\frac{R(x_1,x_2)\mp\sqrt{R(x_1)R(x_2)}}{(x_1-x_2)^2} \tag{2.2}$$

$$x_{1,2}=p\pm iq,\quad R(z)=-z^4+6hz^2+4\mu lz+\mu^2-k^2$$
$$R(x_1,x_2)=-x_1{}^2x_2{}^2+6hx_1x_2+2\mu l(x_1+x_2)+\mu^2-k^2$$

It is obvious that $x_2=\bar{x}_1$, $x_1=\bar{x}_2$ (here the overbar denotes the complex conjugate). Since $R(x_1, x_2)$ and $(x_1-x_2)^2$ are symmetric polynomials in $x_1$ and $x_2$ with real coefficients, they take real values only. Further, the expression

$$R(x_1)R(x_2)=R(x_1)R(\bar{x}_1)=R(x_1)\overline{R(x_1)}$$

is obviously nonnegative. The realness of variables $s_1$, $s_2$ now follows from formula (2.2).

From formula (2.1) it follows that the region of real motions is determined by the inequalities $\Phi(s_1) \leqslant 0$, $\Phi(s_2) \leqslant 0$. Figure 1 shows the region of possible motions when the polynomial $\Phi(z)$ has five real roots (it is unhatched). Motion cannot take place in regions *3*, *5* and *7* since points $s_1$ and $s_2$ such that $s_1 = s_2$ exist within these regions. It then follows from (2.2) that $R(x_1) = \overline{R(x_2)} = 0$. Since $R(z)$ is a fourth-degree polynomial, the equation $R(z) = 0$ can have no more than four roots for fixed constant first integrals. Therefore, no more than four points at which $s_1 = s_2$ exist on the invariant tori. But there are infinitely many such points in regions *3*, *5*, *7*.

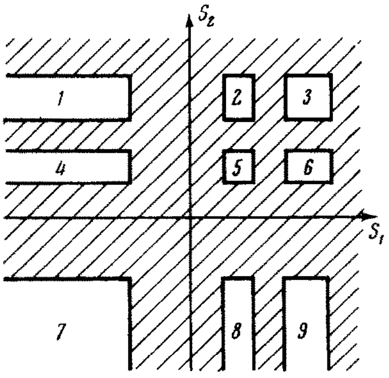

Fig. 1

Thus, motion can take place only in regions *1*, *2*, *4*, *6*, *8*, *9*. In order to study this motion we rewrite Eq. (2.1) as

$$\frac{ds_1}{dt} = \frac{\sqrt{-\Phi(s_1)}}{2(s_1 - s_2)}, \qquad \frac{ds_2}{dt} = \frac{-\sqrt{-\Phi(s_2)}}{2(s_1 - s_2)} \tag{2.3}$$

Let the initial conditions for $s_1$, $s_2$ lie in one of the six regions indicated and let both radicals in (2.3) be positive at the initial instant. For definiteness we assume that $s_1 > s_2$ (i.e. motion occurs in regions *6*, *8*, *9*). Then, at the succeeding instants $s_1$ increases, while $s_2$ decreases. This will go on until $s_1$ (or $s_2$) reaches a root of the polynomial $\Phi(z)$ or goes to infinity. Note that $s_1$ (or $s_2$) goes to infinity in finite time. This follows from the convergence of the integral

$$\int_{-\infty}^{a} \frac{z\,dz}{\sqrt{-\Phi(z)}}$$

where $a$ is the smallest simple root of $\Phi(z)$. For example, let $s_1$ reach a root of $\Phi(z)$ or go to infinity. Then the radical in the first equation of (2.3) changes sign and at the succeeding instants $s_1$ decreases. This goes on once again until $s_1$ (or $s_2$) reaches a root of polynomial $\Phi(z)$ or goes to infinity. And so on.

Let us show that for small values of $\mu$ the real motion takes place in "sleeves" *1* and *9*. At first let $\mu = 0$. We ascertain in which region the initial values for $s_1$ and $s_2$ fall. For $\mu = 0$ the polynomial $\Phi(z)$ is independent of the constant area $(2l)$ and has the form

$$\Phi(z) = z(z - 3h - k)^2 (z - 3h + k)^2$$

The energy integrals and the Kowalewska integral are written thus:

$$H : p^2 + q^2 + r^2 / 2 = 3h, \qquad K : p^2 + q^2 = k \quad (k > 0)$$

It is obvious that on any of the two connected components of set $\{H = 3h,\ K = k^2\}$ in $R^3\ \{pqr\}$ there exist points whose $p$-coordinates equal zero. Let us examine these initial conditions. Then from (2.2) we get that $s_1 = 0$, $s_2 = 3h + k$. Note that the root $(3h - k)$ of polynomial $\Phi(z)$ lies to the right of zero, since $3h - k = r^2 / 2 > 0$. Hence, in this case the region of real motions is $s_1 \leqslant 0$, $s_2 = 3h + k$.

Now let $\mu \neq 0$, but be very small. Then, $s_1$ varies from $-\infty$ up to a number close to zero (since $z = 0$ is a simple root of polynomial $\Phi(z)$ when $\mu = 0$), while $s_2$ is contained between two numbers differing but little from $3h + k$. Consequently, for small values of the parameter $\mu$ real motion takes place in regions *1* and *9*.

We pass on to the computation of the invariants of the dynamic system $(E, g_E{}^t, \nu)$, namely, the gyration numbers of the tangent vector fields on $E$ (which are induced by the Euler-Poisson equations). To be specific we examine the two-dimensional invariant tori which correspond to regions *1* and *9* on the plane $R^2$ $\{s_1, s_2\}$ (or we take the parameter $\mu$ as being small). We denote the roots of polynomial $\Phi(z)$ by $a_0, a_1, a_2, a_3, a_4$; they are arranged in increasing order. Region *1* in Fig. 1 is determined by the inequalities

$$-\infty < s_1 \leqslant a_0, \; a_3 \leqslant s_2 \leqslant a_4$$

In Eqs. (2.3) we make a change of variables $s_1 = s_1(x)$, $s_2 = s_2(y)$ by formulas

$$x = \frac{\pi}{\tau_1}\int\limits_{s_1}^{a_0}\frac{ds}{\sqrt{-\Phi(s)}}, \qquad y = \frac{\pi}{\tau_2}\int\limits_{a_3}^{s_2}\frac{ds}{\sqrt{-\Phi(s)}}, \qquad \begin{array}{l} s_1 \in (-\infty, a_0] \\ s_2 \in [a_3, a_4] \end{array} \tag{2.4}$$

$$\tau_1 = \int\limits_{-\infty}^{a_0}\frac{ds}{\sqrt{-\Phi(s)}}, \qquad \tau_2 = \int\limits_{a_3}^{a_4}\frac{ds}{\sqrt{-\Phi(s)}}$$

Then, $x, y \; (\in [0, 2\pi])$ are angle variables on the invariant tori $T^2(6h, 2l, k^2)$, corresponding to the regions of form *1* under the Kowalewska replacement (2.2). In the new variables $x, y \bmod 2\pi$ Eqs. (2.3) reduce to the form

$$x^{\cdot} = \frac{\pi}{2\tau_1}\frac{1}{s_2(y) - s_1(x)}, \qquad y^{\cdot} = \frac{\pi}{2\tau_2}\frac{1}{s_2(y) - s_1(x)} \tag{2.5}$$

where the $s_i(z)$ are real hyperelliptic functions with period $2\pi$, determinable from relations (2.4). Equations (2.5) have an integral invariant with density $F(x, y) = s_1(x) - s_2(y)$; this function does not vanish anywhere. From (2.5) it follows that the gyration numbers of the dynamic system $(T^2, g^t_{T^2}, \nu)$ equal $\gamma = \tau_2 / \tau_1$. Hence, the gyration numbers of dynamic systems on the invariant tori of Kowalewska's problem equal the ratios of the periods of the hyperelliptic integral

$$\int\limits_{z_0}^{z}\frac{dz}{\sqrt{-\Phi(z)}}$$

where $\Phi(z)$ is the Kowalewska polynomial.

By Liouville's integrability theorem [4], differential equations on $T^2(6h, 2l, k^2)$, defining the dynamic system $(T^2, g^t_{T^2}, \nu)$, reduce in certain angle variables $\varphi_1$, $\varphi_2 \bmod 2\pi$ to the following form:

$$\varphi_1^{\cdot} = \omega_1, \; \varphi_2^{\cdot} = \omega_2; \; \omega_i = \mathrm{const}, \; \omega_1 / \omega_2 = \gamma$$

Consequently, the dynamic system $(T^2, g^t_{T^2}, \nu)$ is completely determined by one invariant, namely, the gyration number $\gamma = \omega_1 / \omega_2$.

Note 1. For any dynamic system on a two-dimensional torus of the form $\varphi_i^{\cdot} = \omega_i \; (i = 1, 2)$ there are in fact an infinite number of gyration numbers, but they are all expressed in terms of the one $\gamma = \omega_1 / \omega_2$ by means of the relation

$$\Gamma = \frac{a\gamma + b}{c\gamma + d}, \text{ where } \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$$

is a rectangular matrix. In particular, $1 / \gamma$ is also a gyration number.

For the tori corresponding to region $1$ in Fig. 1 the gyration numbers are given by formula (2.6). We note that the gyration number for region $9$ is the same as for region $1$. Hence, for small $\mu$ the dynamic systems arising on two connected components of set $E$ are isomorphic.

Note 2. The gyration numbers of vector fields on the two-dimensional invariant tori of the Euler-Poinsot problem are computed in [5] and some of their properties were indicated there. In the Lagrange-Poisson case the gyration numbers equal the ratio of the period of variation of the nutation angle to the period of the mean natural rotation.

**3. On the reduction of differential equations on a torus.** The system of equations (2.5) can be brought to a yet more simple form. Since equations of this form are frequently encountered in investigations of integrable dynamic systems, we examine the general case of such equations specified on an $n$-dimensional torus $T^n$

$$q_i^{\cdot} = \lambda_i / F(q_1, \ldots, q_n), \quad i = 1, \ldots, n \tag{3.1}$$
$$\lambda_i = \text{const}, \quad F = f_1(q_1) + \ldots + f_n(q_n); \quad F > 0 \ (< 0) \ \text{on} \ T^n$$

Without loss of generality we can take all the $\lambda_i$ as nonzero.

Theorem. If $f_i(q_i)$ $(i = 1, \ldots, n)$ are continuous functions, then system (3.1) is reduced to the form

$$\varphi_i^{\cdot} = \lambda_i / \Lambda, \qquad i = 1, \ldots, n$$
$$\Lambda = \frac{1}{(2\pi)^n} \oint_{T^n} F(q_1, \ldots, q_n)\, dq_1 \ldots dq_1 = \sum_{i=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} f_i(x)\, dx$$

by a differentiable change of variables.

To prove this it is sufficient to verify that one such change of variables is the following:

$$\varphi_i = \frac{\lambda_i}{I} \sum_{j=1}^{n} \frac{1}{\lambda_j} [F_j(q_j) - I_j q_j] + q_i$$

$$F_j(t) = \int_0^t f_j(x)\, dx, \qquad I_j = \frac{1}{2\pi} F_j(2\pi), \quad I = I_1 + \ldots + I_n$$

This theorem is applicable to Eqs. (2.5) and yields the following result: a change of variables exists, leading the equations to the system

$$u^{\cdot} = \frac{\pi}{2\tau_1 \Lambda}, \qquad v^{\cdot} = \frac{\pi}{2\tau_2 \Lambda} \tag{3.2}$$
$$\Lambda = \frac{1}{2\pi} \left( \int_0^{2\pi} s_1(x)\, dx - \int_0^{2\pi} s_2(y)\, dy \right) \quad (\Lambda > 0 \ \text{or} \ < 0)$$

Here $\tau_i$ $(i = 1, 2)$ are the periods of the hyperelliptic Kowalewska integral. The given transformations consist only of algebraic operations, of the computation of integrals of known functions, and of the inversion of these integrals. Thus, Eqs. (3.2), defining conditionally-periodic motion of two-dimensional invariant tori, are those same equations which should exist by Liouville's integrability theorem.

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Translated by N. H. C.

UDC 531.31

## ON QUANTITATIVE ESTIMATES OF THE INDETERMINACY OF MOTION

PMM Vol. 39, № 1, 1975, pp. 30-35
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(Received October 8, 1973)

We propose quantitative estimates of the indeterminacy of the prediction of the motion of a controlled system described by ordinary differential equations.

**1.** The motion of a controlled system is specified by the equation

$$dx / dt = F(t, x, u(t)), \quad t \in I, \qquad I = \{t : t_0 \leqslant t < t_*\} \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_r)$ are vectors in real $n$- and $r$-dimensional spaces $R_x^n$ and $R_u^r$, respectively, $t$ is time, $t_0$ is a number, $t_*$ is either a number or the symbol $\infty$. The norm $\|x\| = \max_i |x_i|$ is defined in space $R_x^n$. The vector $x = x(t)$, $t \in I$ characterizes the state of the controlled system, $u = u(t)$, $t \in I$ is an input whose graph $\omega = \{(t, u): u = u(t), t \in I\}$ belongs to an admissible set $\Omega = \{\omega\}$, $\omega\, [t_0, t]$ is the restriction of $\omega$ onto $[t_0, t] \cap I$.

Suppose that a solution of system (1.1), starting on a given open set $V_0 \subset R_x^n$, exists for all $t \in I$ at arbitrarily chosen $\omega \in \Omega$; $x(t) = \varphi(t, t_0, x_0, \omega\, [t_0, t])$, $t \in I$, is any such solution, where

$$x(t_0) = x_0 \in V_0 \tag{1.2}$$

Let $S_\delta(x_0) = \{b_0 : \|b_0 - x_0\| \leqslant {}^1/_2\, \delta\}$, $\delta > 0$ be a ball which characterizes the region of admissible initial states of system (1.1), if the possible measurement errors of the controlled system's initial state are taken into account. At an instant $t \in I$ we consider the set $S_{\rho_t}$ of states of system (1.1) on all possible motions of it (on the graphs of the solutions in $I \times R_x^n$) starting from $S_\delta(x_0)$ for a specified $\omega \in \Omega$. In other words, $S_{\rho t} = \Phi_t(S_\delta(x_0))$ is the image of the ball $S_\delta(x_0)$, where $\Phi_t$ is a mapping